How Secondary Two Express Stream Students Used Algebra and the Model Method to Solve Problems[1]

Ng Swee Fong

National Institute of Education, Nanyang Technological University, Singapore

Abstract: *In Singapore, primary children are taught to use the model method to solve challenging word problems, which otherwise would require algebra to solve. This study aimed to investigate whether learning the model method has any effect on how such students solve word problems after they have learned formal algebra in secondary mathematics. Also students' perceptions of the model method were ascertained. One hundred and forty-five secondary two express stream students took part in this study. The students sat for a four-item test. The study found that the express stream students preferred algebra to the model method. However students had difficulties identifying how unknowns were represented in the model method. Although students had positive perceptions of the model method they were also able to identify its limitations.*

Introduction

Because primary school pupils had difficulties solving challenging word problems, the model method was introduced into the Singapore primary mathematics curriculum to help these pupils in their problem solving activities (Kho, 2000). In his paper 'Mathematical Models for Solving Arithmetic Problems' presented at the 1987 ICMI-SEAMS, Kho described the various types of models and highlighted the efficacy of the model method.
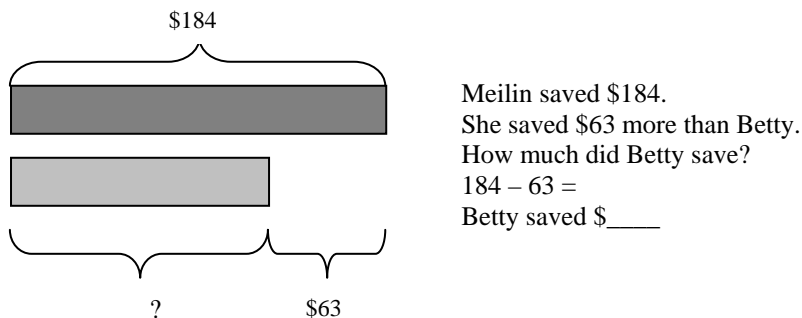
> Without using the model, one may have to resort to the algebraic method to solve structurally complex problems. However, the 'model' method is less abstract than the algebraic method and can be introduced to pupils before they learn to solve algebraic equations. Indeed the models serve as good pictorial representations of algebraic equations. (p.349)

The model method was systematically introduced into the primary four mathematics textbook published in 1983 (Primary Mathematics 4A, 1983). However, by the third edition of the Primary Mathematics series written by the team headed by Kho, the model method was introduced at primary three. The part-whole concept

---

underpinned the model method used to solve word problems at this level. In the following example (CPDD, 1992, 3A, p. 21) pupils were shown how the model method could be used to solve a word problem involving relational information.

$184

Meilin saved $184.
She saved $63 more than Betty.
How much did Betty save?
184 – 63 =
Betty saved $____

? $63

Primary five textbooks (CPDD, 1994) showed how the model method could be used to solve word problems involving rational numbers. In the following example (CPDD, 5A, p. 59) the model method is used to guide pupils in this task.

Mrs Lin made 300 tarts. She sold $\dfrac{3}{4}$ of them and gave $\dfrac{1}{3}$ of the remainder to her neighbour. How many tarts had she left?

300

?

75

?
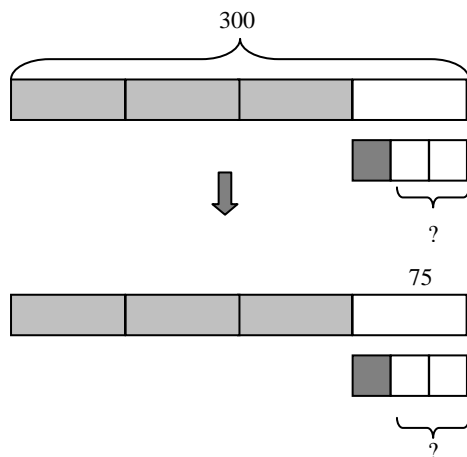
Primary five textbooks (CPDD, 1994) also provided examples showing how the model method could be used to solve word problems that require proportional

reasoning.   Under the topic of ratio, the model method was widely used to solve related problems and the following is one such example (5A, p. 78).

Siti and Mary shared $35 in the ratio 4:3.
How much money did Siti receive?

7 units = $35
1 unit = $
4 units = $
Siti received $ ___
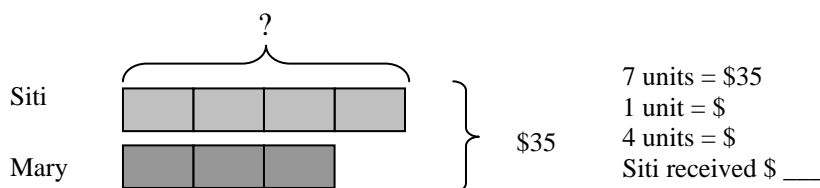
With the aid of the model method, primary six pupils could solve the following Primary School Leaving Examination (PSLE) problem which too requires proportional reasoning.

The participants in a competition are P5 and P6 pupils in the ratio of 2 : 1. All the P5 participants are girls. Among the P6 participants, the ratio of girls to boy is 4 : 3. There are 30 more P5 girls than P6 girls taking part in the competition. How many participants in the competition are girls? (PSLE Mathematics, p. 39)

The model method has been an integral part of the Singapore mathematics curriculum for more than ten years. This paper reports on what effects the learning of the model method has on students in solving algebraic problems.   The above examples show that in Singapore, primary school pupils (9+ to 12+) are taught how to use the model method to solve 'structurally complex problems' (Kho, 2000, 1987). Normally students who have been taught formal algebra but not the model method would solve such problems by constructing a system of linear equations and solving for the unknowns in the equations. However, primary school pupils in Singapore, who have not been exposed to formal algebra, would be expected to draw diagrams, normally made up of rectangles, to model the situation presented in the problem. Mathematically, the rectangles represent the value of the unknown and the pupils are expected to solve for the unknown by analysing the relationships between the rectangles. When these primary school pupils enter secondary school and are taught formal algebra, they now construct systems of linear equations to solve similar problems. Although secondary school mathematics textbooks provide examples linking the system of linear equations with the model method of the primary school, the effort is cursory. Hence, of interest, is what effect, if any, has

the teaching and learning of the model method on the development of algebraic thinking of students who have been introduced to both methods - formal algebra and the model method. This paper addresses the following questions.

    (i)   Which method do secondary two express stream students prefer to use to solve a given problem?

    (ii) Are secondary two express stream students able to recall the model method?

    (iii) Are secondary two express stream students able to identify how unknowns are represented in the model method?

    (iv) What are secondary two express stream students' perceptions of the model method?

The Study

In Singapore, based on their performance in four subjects (Mathematics, Science, English Language and Mother Tongue) in the Primary School Leaving Examination (PSLE), secondary school students are generally divided into three categories – Express Stream, Normal Academic and Normal Technical. Although parents of students with total scores bordering on the first two categories do have a choice as to which stream they prefer their child to continue in, Express Stream students are generally more academically inclined than the other two categories. In this study, two intact classes of secondary two Express Stream students (14+) from two co-educational schools took part. Details of the two groups are presented in Table 1. These four classes were opportunistic samples. These students took the test in May immediately after they have completed their semester one examinations. Sample B took the test one week later than those from Sample A as Sample B completed its semestral examination one week later than Sample A. Students from both samples had completed the topic "Solutions of Equations – Simple Linear Equations and Simultaneous Equations" and this topic formed the bulk of the test in the semestral examination.

Table 1
Students who took part in the study

	Female	Male	Grand Total
Sample A	41	36	77
Sample B	46	22	68
Total	87	58	145

The test (please see Appendix 1) consisted of four items. The students were given an hour to answer the questions in the test. Ten students were then randomly selected from each sample to be interviewed. These interviews were conducted

immediately after the test. The interviews were meant to provide greater insight into how students actually solved the problems. Each interview took about 15 minutes and was tape-recorded for analysis. Pseudonyms are used to identify students who took part in the study. On the average, students in Sample B had performed better than students in Sample A in the PSLE.

The Test Instrument
There were four items in the test. Students could use any method to solve Item 1. The objective of Item 1 was to determine the students' preferred method of solution. There were two parts to Item 2. Part (a) consisted of a problem which was accompanied by a model method solution. Students were asked to determine whether or nor the model was a logical representation of the problem. If they thought the model method was correct, they were asked to use the model method to solve part 2. The objective of Item 2 was to ascertain if the students could still recall and apply the model method. A model method solution was provided for the problem presented in Item 3. Here students had to identify what were the unknowns in the problem and how the unknowns were represented in the model method. The final item asked students to state, in not more than 5 sentences, what they thought of the model method.

Findings and discussion

The findings from the two samples have not been merged so as to maintain the individuality of the two samples. As this was not a comparative study, whatever differences that have arisen between the two samples have been noted but not discussed. The intent was that such differences could be a motivation for more in-depth study to try and understand how different students use the model method and algebra to solve problems. Students' written responses to each item are discussed in turn and, where appropriate, their interviewed responses are used to give greater meanings to the written responses. The answer to the related research question is provided in the concluding section of each item.

Item 1
Research question: *Which method would students use to solve a given problem?*
Students could use any logical method to solve Item 1 and the 5 different approaches identified are shown in Table 2.

Using Algebra
About 60% of the students used algebra to solve Item 1. This was probably due to the fact that the bulk of the semester one curriculum time was spent on the topic "Solutions of Equations – Simple Linear Equations and Simultaneous Equations"

Table 2
Students' Response to Item 1

	Algebra	Model and algebra	Listing	Grouping	Model	Wrong or no solution	Total
Sample A	36	5	13	16	4	3	77
Sample B	48	4	5	11	0	0	68
Total	84	9	18	27	4	3	145
%	57.9	6.2	12.4	17.9	3.5	2.1	100

and the semester one mathematics paper was also focused on this topic. Of the 'algebra' students, 82.1 % formed equations of one unknown, usually in x, and they solved for the unknown. The remaining 17.9% formed simultaneous equations in two unknowns and solved for the appropriate unknown (see Figure 1). More students from Sample B than from Sample A used algebra to solve Item 1. Responses from the interviewed students provided some insight into why algebra was the method of choice. Some of the students did not know how to draw the model for this problem as explained by two of the students.

> Rama: I used algebra because I did not know how to draw models.

> Cai Cai: I don't know the model method quite well. I remember the algebra because I learnt it this year.

Others explained that they were more familiar with algebra as the method was fresh in their minds and also, they were encouraged by the teacher to use algebra

> Leeta: I recently learnt the algebraic method and I think it is quite convenient as it saves me more time than the model method. I am used to this method.

> Mei Liang: Used algebra rather than model. My teacher said it's not very good to use model, so perhaps it's because he wants us to get used to algebra.

Still others thought that algebra was less tedious than model drawing. Moreover with algebra they could check if their solutions were correct.

Mei Lin: I didn't use the model method because I thought algebra would be faster.

Zhi: Used algebra to solve this problem because algebra is easier to understand. And you can get the answer immediately. Can't get the answer immediately with model. With models you need to do some more thinking before you can get the answer. So it's harder.

Yee Tee: I find model too tedious, time-consuming and might not be accurate most of the time. Model have to be drawn carefully and must be detailed. If I left out one detail, I don't think I can still find the answer. Algebra, I find it more easy if you know the concepts. Then even though you can still check back. But with model, I don't think you can check back for most of the questions.

Figure 1. Algebraic equations of one and two unknowns

Using a combination of model and algebra

Nine (6.2%) students used a combination of algebra and model method to solve this problem. They drew the models to represent the situation and then used the models to help them form the algebraic equations. Figure 2 shows an example where a
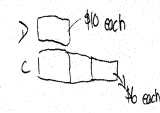
Using algebra

Using algebra and model

Using grouping

Using listing

Figure 2. Methods used by students to solve Item 1

student's algebraic solution was supplemented by a model drawing. All nine students formed linear equations of one unknown. Two of those interviewed said they used the model to help them visualise the problem.

Using Listing
Eighteen (12.4%) students listed the possible combinations and solved the problem this way. Three of the eighteen students supplemented their solutions with models (see Figure 2). Two of those interviewed explained that they used the "guess and check" method because they "don't know how to use algebra". Thirteen of the eighteen students who used listing to solve Item 1 came from Sample A.

The Model Method
Four students, all from Sample A chose the model method to solve Item 1. Two of those interviewed explained that the models helped them visualise the problem situation.

Grouping Method
Twenty-five students, with about the same number of students from each sample, used the grouping method to solve Item 1. The students who were interviewed coined the term 'grouping'. Of all the methods used to solve Item 1, the grouping method was most interesting as it showed students using the model method in a novel way. In the standard model solution, the rectangles, known as parts or units, were used to represent the unknown and solving for the value of the parts would mean solving for the unknown. However, students who used the grouping method did not see the rectangles as unknowns. The rectangles had values. In this case the rectangle representing the ducks had a value of $10 and the rectangles representing the chickens each had a value of $6. The total value of the model was $28. Students then worked out how many such groups of $28 there were in $420. Students who used the grouping method gave a very concise solution to Item 1.

Two students explained why they used grouping to solve Item 1. For student Siew Wan, the model method influenced his solution. For Ying Kai the tedium of drawing models influenced her not to draw the models but rather to use her own construction of meaning of groups to help her solve the problem. She saw her method as being superior to algebra.

> Siew Wan: Used groups to solve the problem, models influence my solution.

> Ying Kai: Usually in a formal exam, you would have to sketch out the graph, and then it'll be very untidy, so you'll have to use a ruler and it's

going to take a lot of time and if you anyhow sketch, you'll have to erase them all after that, so it is wasting a lot of time. So I'd rather use my own way by just grouping them into groups is much better. Didn't choose to use algebra because if you can group them is faster.

Which method would students use to solve a given problem? Item 1 shows that secondary two express stream students who were taught algebra, when given a choice, would prefer to use algebra over other methods to solve a given problem. Also students who could re-conceptualise the model method could produce more sophisticated methods of solution than even that of algebra. It could be argued that students who used grouping to solve Item 1 exhibited a better understanding of what was required of them than those who used algebra. Using algebra procedurally, students eventually came to this point:

$$28x = 420$$
$$x = 15$$

Students who used groups came to the same point but they knew they were looking for the number of groups of 28. Were students who used algebra to solve Item 1 aware that while x represented the number of ducks, they were also solving for the number of groups of 28 and that x could represent the number of groups of 28?

Item 2
Research question: *Were students able to recall the model method?*
Students' written responses showed that all but five thought that the solution to part 1 was correct and applied the same method to solve part 2. Three other students did not disagree with the model method solution but they chose to use methods such as grouping (1 student) and algebra (2 students) to solve part 2 of Item 2. This suggested that students could still recall the model method and were able to apply the method to solve a similar problem. The five students who used another method to solve part 2 did not explain why they thought the model method was wrong. No reason could be provided as to why they thought the model method was wrong as they were not part of those randomly selected to be interviewed.

Table 3
Students' response to Item 2

	A	B	C	Total
School A	72	1	4	77
School B	65	2	1	68
Total	137	3	5	145
%	94.5	2.1	3.4	100

A: Used model method to solve part 2
B: Used another method to solve part 2
C: Model method was wrong and used another method to solve part 2

Item 3

Research question: *Were students able to identify how unknowns are represented in the model method?*

A model method solution was provided for the problem presented in Item 3. Here students had to identify what were the unknowns in the problem and how the unknowns were represented in the model method.

Table 4
Students' Response to Item 3

	A	B	C	D	Total
School A	35	29	12	1	77
School B	32	31	4	1	68
Total	67	60	16	2	145
%	46.2	41.4	11.0	1.4	100

A: Answered both parts correctly
B: Answered (a) correctly but incorrect for (b)
C: Answered both parts incorrectly
D: Answered (a) incorrectly but (b) correctly

Sixty-seven (46.2%) students were able to identify the unknowns in part (a) and were able to state how the unknowns were represented in the model method. These students identified the unknowns in the part (a) as the number of books Jack and Jill each had. They used words such as "bars", "boxes", "rectangles", "lengths of boxes", "units or boxes" to identify the unknowns in the model method. The following are some students' written responses to part (b).

Sher Liang: They are represented by rectangles, each different sized rectangle represents a different number.

Sing Yee: The unknowns are represented by the length of the rectangular boxes.

Pei Pei: They are represented by the block belonging to Jill becomes smaller after she lost 3 books. The 'block' belonging to Jack is doubled when the number of books he have is doubled.

About 41% of the students were able to answer part (a) correctly but were unable to identify how the unknowns were represented in the model method. It was very common to find students writing that the unknowns were represented by the models. The following are some of the written responses of students.

Kian Sing: They are represented by diagrams.

Joh: They are represented by models.

David: Unknowns are represented by a question mark or an assumption of unit is used and no figures are stated.

Lisa: Represented by ratio.

The interviewed responses gave further insight into students thinking of the model method and unknowns. Kian Sing explained that the whole picture could represent the unknown. Others explained that they were unsure how unknowns were represented in the model method.

Overall, seventy-six (about 52%) students were unable to say how the unknowns were represented in item 3. Most students would have been formally taught how to use the model method to solve challenging word problems from Primary 4 onwards. The fact that more than half the students in this study were unable to identify how the unknowns were represented in the model method suggests that perhaps the idea that the rectangles represented unknowns of a problem was not made clear to students. Often students said that the units represented the unknown in the problem but were unable to say what the units were. Those who gave wrong written responses to parts (a) and (b), when interviewed, explained that they did not know what an unknown meant. This suggests that there is a need to make explicit the meaning of unknowns and how they are represented in the models drawn to represent the problem situation.

Item 4
Research question: *What were students' perceptions of the model method?*
In response to Item 4, one hundred fourteen of the written responses showed that students found the model method useful as it helped them to 'visualise' the problem. However all qualified the usefulness of the model method in that the model method was time consuming. In order to draw the models, all information provided in the

Table 5
Students' responses to Item 4

	Visualise the problem	Did not like models- preferred algebra	No response	Total
School A	72	4	1	77
School B	42	24	2	68
Total	114	28	3	145
%	78.6	19.3	2.1	100

question had to be identified and included in the representation. Should a detail be missing, it would not be possible to solve the problem. In comparison, algebra was a better method than the model method as algebra was more efficient. Some students explained that the model method was useful only when the numbers were small. Some of the students' responses are listed below.

> Ele: I think both are important. Models are good in some ways because in some questions it requires you to see the proportion. I think models are more effective. I think algebra is more accurate and if you do it properly, you can get it correct.

> Mei: The model is very easy to visualise, when you draw out. Easier to solve problems, but it is a waste of time in exams. Algebra is faster

> Pei Pei: The model problem helps you to see the problem clearer, to some extent. When the number is too big, the model is of not much use. But if it's small, it helps you see the problem.

> Ying Kai: For weaker students, I feel that it is much easier to use the model because you can let them visualise if they don't understand the question. But if you understand the question I feel you should zoom right ahead with algebra.

> Hong Sheng: It gives us a good imagery of a question.

Twenty-eight students preferred algebra to the model method.

> Maggie: Model approach is useful as it makes understanding of the problem easier. But, at times, it is quite difficult to draw out a model of the problem and thus, when you meet a problem, models is (sic) not actually helpful. Although I find the model approach useful, I think using algebraic solving is easier and even more useful.

> Zhi Zhi: I think the model are good (sic) but not as good as algebra, because algebra can give you a direct approach, but the model you still have to think more about it, then you can see the picture. Sometimes I think model complicates the whole thing. Algebra is simpler.

> Cathy: The model approach is easier for the students to master and use when compared to algebra, which requires mastery of the formula concerned with the equation. Therefore, the model approach is easier for Primary students to learn. Although algebra can be used more extensively when finding out 2 unknowns, and the model approach is rather limited.
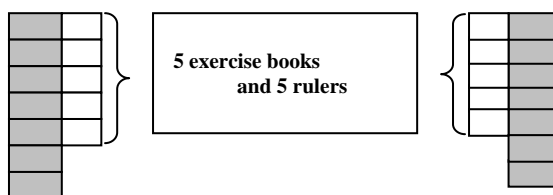
Implications for teaching

This study shows that while most students retained knowledge of the model method as a method for problem solving, once they were taught algebra, algebra became the preferred method of solution for problems presented in the test. While students knew the efficacy of the model method, they also knew its limitations. More importantly, students were not clear how the unknowns were represented in the model method. Perhaps primary teachers, when teaching pupils the model method, likely need to be more explicit that the rectangles are used to represent the unknowns as presented in the word problems.

Teaching primary pupils the model method is an expectation of the Singapore curriculum. Although many pupils have developed the skills and competence needed in using the model method to solve algebraic problems, they also need the experiences that link the model method with the algebraic representation. Without this link being made overt, the benefits of the model method are often not fully utilised. One way to form this link is to have students, in the introductory course to formal algebra, construct parallel representations for the problems. Students first draw the model method for the problem and then record those steps that correspond to the diagrammatic representation of the problem as presented in the model. In this way, students may have a clear sense that the algebraic expressions are the abstract representation of that which is being modeled by the rectangles in the model method. Fong (1994) has provided many such examples. The following example demonstrates how the link can be made with problems involving two unknowns.

Evonne paid $8.10 for 7 exercise books and 5 rulers. The total cost of an exercise book and a ruler was $1.30. Find the cost of the exercise book and the ruler.

Grey box represents the cost of one exercise book and the white box, the cost of one ruler.

How the model serves as a link between the visual and the symbolic representation.

7 exercise books and 5 rulers cost $8.10
5 exercise books and 5 rulers cost
  $1.30 x 5 = $6.50
 	
Therefore 2 exercise books cost
  $8.10 - $6.50 = $1.60
1 exercise book costs 80¢.
A ruler would cost 50 ¢.
	Let e be the cost of one exercise
	book and r the cost of one ruler.
	
	7e + 5r = $8.10 (1)
	e + r = $1.30 (2)
	5e + 5r = $6.50 (2) x 5
	(1) – (2) x 5
	2e = $1.60
	1 exercise book costs 80¢.

Anecdotal evidence provided by school teachers and students suggests that secondary students dislike algebra. Perhaps if secondary school teachers emphasize the parallel presentation of problems, students, instead of viewing algebra as a study of routinized manipulations and learned algorithms, may view the subject as more meaningful and see the operations connected to the model method. Moreover, as students found the model method useful in helping them visualize problem, teachers could show students how they can turn to the model method when confusions arise with the abstractions.

Further Research

This study showed that secondary two express stream students preferred algebra to the model method. However, Fong (1994) suggested that there was anecdotal evidence to suggest that some lower secondary students, although they had been taught algebra, continued to use the model method to solve problems. The teachers whose students took part in this study explained that they found those students in the normal academic and normal technical streams preferred the model method to algebra. Research is needed to ascertain if this perception is accurate.

Research (e.g. Küchemann, 1981) has shown that for students to have any real understanding of algebra, they must at least see that the letter stands for a specific unknown. If students were presented with the model method solution to a problem, would students be able to use the model method to form the parallel algebraic equations to problem? Would they be able to identify the specific unknowns represented by the rectangles with those in the algebraic equations? More research is needed to understand how students link the unknowns represented by the rectangles with the unknowns of the algebraic equations.

References

CPDD (1992). *Primary mathematics 3A: Third Edition.* Singapore: Federal Publications.

CPDD (1994). *Primary mathematics 5A:* Third Edition. Singapore: Federal Publications.

Fong, H. K. (1994). Bridging the gap between secondary and primary mathematics. *Teaching and Learning, 14*(2), 73 - 84

Kho, T. H (1987). Mathematical models for solving arithmetic problems. *Proceedings of the Fourth Southeast Asian Conference on Mathematical Education (ICMI-SEAMS). Mathematical Education in the 1990's.* June 1 – 3, (pp. 345 – 351). Singapore: Institute of Education

Kho, T. H. (2000). Personal Communication

Küchemann, D. E. (1981). Algebra. In K. M. Hart (Ed.) *Children's understanding of mathematics: 11 – 16.* London: Murray.

Primary Mathematics 4A. (1983). The Curriculum Planning & Development Division, Ministry of Education, Singapore: Federal Publications.

PSLE Mathematics (1996 – 2000). *Mathematics – PSLE questions 1996 – 2000.* Reproduced with the permission of the Ministry of Education, Singapore: Hillview Publications Pte. Ltd.

Author:

Ng Swee Fong, Assistant Professor**,** National Institute of Education, Nanyang Technological University, Singapore. ngsf@nie.edu.sg

Appendix 1

Test Instrument
Item 1
A man has some ducks and three times as many chickens. He sold the ducks at $10 each and chickens at $6 each. If his taking from the sale of all his animals came to a total of $420, how many chickens did the man have?

Item 2
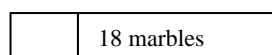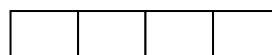Read the solution to the following question.
Linda and Paul shared some marbles in the ratio of 3:2. (a) In a game, Paul lost half of his marbles to Linda. After the game, Linda had 18 more marbles than Paul. How many marbles did Linda have after the game?

Before   **After**

Linda

Paul

18 marbles

3 parts = 18 marbles
4 parts = 24 marbles
Linda had 24 marbles.

If you think the solution is correct, show your understanding by solving the following problem using a similar approach.

In Peter's class, the ratio of girls to boys is 3:2. If the number of boys were doubled and 6 girls added, there would be an equal number of boys and girls. How many students were in the class at the beginning?

Item 3
A model method solution is provided for Item 3.

The sum of the number of books Jack and Jill have is 20. If Jill lost 3 of her books and Jack doubled the number he has, they would then have a total of 30 books. How many books does each have?

Solution
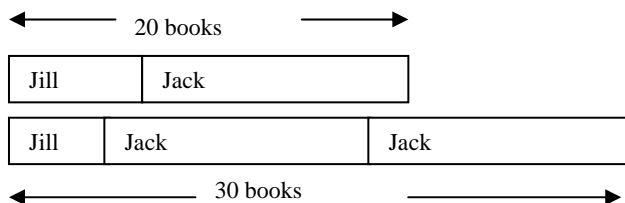
Jack has 13 books. Jill has 7 books.
   (a)   What are the unknowns in this problem?
   (b)   In the model method solution, how are the unknowns represented?

Item 4
Write, in not more than 5 sentences, what you think of the model method.